

# WELL-POSEDNESS FOR THE CAUCHY PROBLEM OF SPATIALLY WEIGHTED DISSIPATIVE EQUATION

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**ABSTRACT.** This paper mainly investigates the Cauchy problem of the spatially weighted dissipative equation with initial data in the weighted Lebesgue space. A generalized Hankel Transform is introduced to derive the analytical solution and a special Young's Inequality has been applied to prove the space-time estimates for this type of equation.

## 1. INTRODUCTION

We consider the Cauchy problem of the following spatially weighted dissipative equation

$$(1) \quad \begin{cases} \partial_t u - |x|^\beta \Delta u = \pm |u|^b u & (x, t) \in \mathbb{R}^n \times [0, \infty), \\ u(x, 0) = u_0(x) & x \in \mathbb{R}^n. \end{cases}$$

Nowadays, the diffusion equations with variable coefficients have wide application in physics, chemistry and engineering etc., and attract more attention. As far as we know, there is few literature on the well-posedness and space-time estimates of such type of dissipative equation. In [8], Miao studied the general parabolic type equation where the diffusion operator  $A = \sum_{|\alpha| \leq 2m} a_\alpha(x) \partial^\alpha$  is strictly parabolic type. A unified method based on the space-time estimates has been introduced to demonstrate the well-posedness result. It is worth noticing that  $|x|^\beta \Delta$  does not absolutely satisfy the strict parabolic condition. Compared with the standard Heat equation, or the fractional dissipative equation(see [9],[18]), the spatial weight prevents us from applying the partial Fourier transform. As a result, we need to explore new approaches.

In this paper, we aim to solve this issue by introducing a special Hankel transform. In accordance with the coefficient, we call it  $\beta$ -Hankel transform. As we know, the standard Hankel transform is a natural generalization of the Fourier transform of radial functions. It is closely related with the Bessel operator and has some nice properties such as  $L^2$  isometry and self-adjointness, etc. [2],[5],[6],[7],[11]. Hankel transform has

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widely applications in the study of PDEs, especially for the radial solutions of dispersive equations. For instance, in [1][12] F. Planchon et al. applied the Hankel transform to obtain the Strichartz estimates for the wave equation with inverse square potential. In [13], Tao gave the double end-point Strichartz estimate of the Schrödinger equation. In [3], Chen studied the similar Schrödinger equation with inverse square potential. Recently, Miao [10] obtained the maximal estimate to the the Schrödinger equation with inverse square potential. All the above work used the Hankel transform to get the explicit solution formula.

Before we state the main result, we first introduce the definitions of admissible and generalized triplets and give the functional space we use.

**Definition 1.1.** *The triplet  $(m, p, q)$  is called an admissible triplet (for the  $k$ -th model) if*

$$\frac{1}{m} = \frac{n - \beta + 2k}{2 - \beta} \left( \frac{1}{q} - \frac{1}{p} \right),$$

where

$$1 < q \leq p < \begin{cases} \frac{q(n-\beta+2k)}{n+2k-2}, & \text{for } n > 2 - 2k; \\ \infty, & \text{for } n \leq 2 - 2k. \end{cases}$$

**Definition 1.2.** *The triplet  $(m, p, q)$  is called a generalized admissible triplet (for the  $k$ -th model) if*

$$\frac{1}{m} = \frac{n - \beta + 2k}{2 - \beta} \left( \frac{1}{q} - \frac{1}{p} \right),$$

where

$$1 < q \leq p < \begin{cases} \frac{q(n-\beta+2k)}{n+2k-2q+(q-1)\beta}, & \text{for } n > 2q + (1 - q)\beta - 2k; \\ \infty, & \text{for } n \leq 2q + (1 - q)\beta - 2k. \end{cases}$$

Here  $k$  is a positive integer associated with the  $k$ -th model which will be introduced in section 2.

**Remark 1.3.** (i) *One can easily find that for the given  $\beta$  and  $k$ ,  $m$  is unique determined by  $p$  and  $q$ . Usually we write  $m = m(p, q)$ .*

(ii) *It is easy to see that  $q < m \leq \infty$  if  $(m, p, q)$  is an admissible triplet. The condition  $q < m$  is required from the application of Marcinkiewicz interpolation theorem in Lemma 3.2.*

(iii) *It is easy to see that  $1 < m \leq \infty$  if  $(m, p, q)$  is a generalized admissible triplet.*

Now we define the  $\mathcal{L}$ -type space as

$$X(I) := C(I; L_{d\eta}^q(\mathbb{R}^+)) \cap L^m(I; L_{d\eta}^p(\mathbb{R}^+)),$$

and  $\mathcal{C}$ -type space as

$$Y(I) := C_b(I; L_{d\eta}^q(\mathbb{R}^+)) \cap \dot{\mathcal{C}}_m(I; L_{d\eta}^p(\mathbb{R}^+)),$$

where  $I = [0, T)$  for  $T > 0$ . And the weighted Lebesgue space  $L_{d\eta}^p(\mathbb{R}^+)$ , time-weighted space-time Banach space  $\mathcal{C}_\sigma(I; L_{d\eta}^q(\mathbb{R}^+))$  and the corresponding homogeneous space  $\dot{\mathcal{C}}_\sigma(I; L_{d\eta}^q(\mathbb{R}^+))$  are defined as follows,

$$L_{d\eta}^p(\mathbb{R}^+) := \left\{ f \in \mathcal{S}'(0, \infty) \left| \|f\|_{L_{d\eta}^p(\mathbb{R}^+)}^p = \int_0^\infty |f(r)|^p d\eta(r) < \infty \right. \right\};$$

$$\begin{aligned}\mathcal{C}_\sigma(I; L_{d\eta}^q(\mathbb{R}^+)) &:= \left\{ f \in C(I; L_{d\eta}^q(\mathbb{R}^+)) \left| \|f; \mathcal{C}_\sigma(I; L_{d\eta}^q(\mathbb{R}^+))\| = \sup_{t \in I} t^{\frac{1}{\sigma}} \|f\|_{L_{d\eta}^q(\mathbb{R}^+)} < \infty \right. \right\}; \\ \dot{\mathcal{C}}_\sigma(I; L_{d\eta}^q(\mathbb{R}^+)) &:= \left\{ f \in \mathcal{C}_\sigma(I; L_{d\eta}^q(\mathbb{R}^+)) \left| \lim_{t \rightarrow 0^+} t^{\frac{1}{\sigma}} \|f\|_{L_{d\eta}^q(\mathbb{R}^+)} = 0 \right. \right\}.\end{aligned}$$

Define the norm

$$\|\cdot\|_{X(I)} := \|\cdot\|_{L^\infty(I; L_{d\eta}^q(\mathbb{R}^+))} + \|\cdot\|_{L^m(I; L_{d\eta}^p(\mathbb{R}^+))}$$

and

$$\|\cdot\|_{Y(I)} := \|\cdot\|_{L^\infty(I; L_{d\eta}^q(\mathbb{R}^+))} + \sup_{t \in I} t^{\frac{1}{m}} \|\cdot\|_{L_{d\eta}^p(\mathbb{R}^+)}.$$

At the moment, it is ready for us to introduce the main results. Consider the radial solution  $u(t, r)$  of (1) satisfying

$$(2) \quad \begin{cases} \partial_t u - r^\beta (\partial_{rr} u + \frac{n-1}{r} \partial_r u) = F(u(t, r)), \\ u(0, r) = u_0(r), \end{cases}$$

where the nonhomogeneous term  $F(u) = \pm |u|^b u$ . Let  $k = 0$ ,  $\gamma = \frac{n-\beta}{2-\beta}$  and  $d\eta(r) = r^{n-1-\beta} dr$ , we have the following theorem on the existence of local solutions or global small solutions.

**Theorem 1.4.** *Let  $1 \leq q_0 = \gamma b \leq q$  and  $u_0 \in L_{d\eta}^q(\mathbb{R}^+)$ . Assume  $(m, p, q)$  is an arbitrary admissible triplet with  $k = 0$ .*

(i) *There exists  $T > 0$  and a unique solution  $u \in X(I)$  to the problem (2), where  $T = T(\|u_0\|_{L_{d\eta}^q})$  depends on  $\|u_0\|_{L_{d\eta}^q}$  for  $q > q_0$ .*

(ii) *If  $q = q_0$  then  $T = \infty$  provided that  $\|u_0\|_{L_{d\eta}^q}$  is sufficiently small. In other words, there exists a global small solution  $u \in C_b([0, \infty); L_{d\eta}^q(\mathbb{R}^+)) \cap L^m([0, \infty); L_{d\eta}^p(\mathbb{R}^+))$ .*

(iii) *Let  $I = [0, T^*)$  be the maximal existence interval of the solution  $u$  to the problem (2) such that  $u \in C_b([0, T^*); L_{d\eta}^q(\mathbb{R}^+)) \cap L^m([0, T^*); L_{d\eta}^p(\mathbb{R}^+))$  for  $q > q_0$ . Then,*

$$\|u(t)\|_{L_{d\eta}^q} \geq \frac{C}{(T^* - t)^{\frac{1}{b} - \frac{\gamma}{q}}}.$$

In a similar manner, we can also prove the following well-posedness results under the  $\mathcal{C}$ -space theory.

**Theorem 1.5.** *Let  $\gamma = \frac{n-\beta}{2-\beta}$ ,  $1 \leq q_0 = \gamma b \leq q$  and  $u_0 \in L_{d\eta}^q(\mathbb{R}^+)$ . Assume  $(m, p, q)$  is an arbitrary generalized admissible triplet with  $k = 0$ .*

(i) *There exists  $T > 0$  and a unique mild solution  $u \in Y(I)$  to the problem (2), where  $T = T(\|u_0\|_{L_{d\eta}^q})$  depends on  $\|u_0\|_{L_{d\eta}^q}$  for  $q > q_0$ .*

(ii) *If  $q = q_0$  then  $T = \infty$  provided that  $\|u_0\|_{L_{d\eta}^q}$  is sufficiently small. In other words, there exists a global small solution  $u \in C_b([0, \infty); L_{d\eta}^q(\mathbb{R}^+)) \cap \dot{\mathcal{C}}_m([0, \infty); L_{d\eta}^p(\mathbb{R}^+))$ .*

(iii) *Let  $I = [0, T^*)$  be the maximal existence interval of the solution  $u$  to the problem (2) such that  $u \in C_b([0, T^*); L_{d\eta}^q(\mathbb{R}^+)) \cap \dot{\mathcal{C}}_m([0, T^*); L_{d\eta}^p(\mathbb{R}^+))$  for  $q > q_0$ . Then:*

$$\|u(t)\|_{L_{d\eta}^q} \geq \frac{C}{(T^* - t)^{\frac{1}{b} - \frac{\gamma}{q}}}.$$

**Remark 1.6.** *Our method can be further applied to the diffusion operator with inverse square potential such as*

$$A = |x|^\beta (\Delta + \frac{a}{|x|^2}).$$

*As the technique reason, the coefficient  $\beta$  is restricted on  $[0, 2)$  in this paper. It is worth noticing the case  $\beta = 0$  is reduced to the standard Heat equation.*

The paper is organized as follows. In Section 2, we show some preliminary work. The theory of Hankel transform and the property of Bessel function has been revisited. Then, we introduce  $\beta$ -Hankel transform and its inverse transform. With these definitions, a weighted  $L^2$  isometry is investigated which is similar to the standard Hankel transform. Moreover, in order to obtain the space-time estimates, an associated convolution operator and the Young's inequality are introduced. In Section 3, the semigroup  $S_\mu(t) = e^{t(-|x|^\beta \Delta)}$  is defined to derive explicit solution formula of the  $k$ -th model. Consequently, a detailed analysis of the kernel function is given followed by the space-time estimates of the admissible triplets. Section 4 is devoted to the radial solution of the nonlinear case  $F(u) = |u|^b u$ . The well-posedness results of local solution and the small global solution are given by the contraction mapping technique.

## 2. THE LINEAR $k$ -TH MODEL AND ITS INTEGRAL SOLUTION

In this paper, we always denote:

$$\lambda = \lambda(n) = \frac{n-2}{2}, \quad \mu(k) = \frac{n-2}{2} + k, \quad \text{and} \quad \mu(\beta, k) = \frac{2\mu(k)}{2-\beta}.$$

Here  $n \geq 2$  stands for the dimension of Euclidean space and  $k$  stands for the degree of spherical harmonic subspace. For simplicity, we denote  $\mu = \mu(\beta, k)$ . We start this section from recalling the spherical harmonics expansion. Let

$$x = r\theta \quad \text{and} \quad \xi = \rho\omega \quad \text{with} \quad \theta, \omega \in \mathbb{S}^{n-1}.$$

For any  $g \in L^2(\mathbb{R}^n)$ , we have

$$g(x) = g(r\theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} a_{k,l}(r) Y_{k,l}(\theta),$$

where

$$\{Y_{k,1}, \dots, Y_{k,d(k)}\}$$

is the orthogonal basis of the space of spherical harmonics of degree  $k$  on  $\mathbb{S}^{n-1}$ , called  $\mathcal{H}^k$ , having dimension

$$d(k) = \frac{2k+n-2}{k} C_{n+k-3}^{k-1} \asymp k^{n-2}.$$

We remark that for  $n=2$ , the dimension of  $\mathcal{H}^k$  is independent of  $k$ . Obviously, we have the orthogonal decomposition

$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}^k.$$

By orthogonality, it gives

$$\|g(x)\|_{L^2_\theta(\mathbb{S}^{n-1})} = \|a_{k,l}(r)\|_{l^2_{k,l}} = \left( \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} a_{k,l}^2(r) \right)^{\frac{1}{2}}.$$

Now we consider the following semilinear spacially weighted dissipative equation in polar coordinates,

$$(3) \quad \begin{cases} \partial_t u - |x|^\beta \Delta u = f(x, t) & (x, t) \in \mathbb{R}^n \times [0, \infty), \\ u(x, 0) = u_0(x) & x \in \mathbb{R}^n. \end{cases}$$

Let  $V(t, r, \theta) = u(t, r\theta)$ , the initial data  $V_0(r, \theta) = u_0(r\theta)$  and the inhomogeneous term  $F(t, r, \theta) = f(t, r\theta)$ . Then  $V(t, r, \theta)$  satisfies:

$$\begin{cases} \partial_t V - r^\beta (\partial_{rr} V + \frac{n-1}{r} \partial_r V + \frac{1}{r^2} \Delta_\theta V) = F(t, r\theta), \\ V(0, r, \theta) = V_0(r\theta). \end{cases}$$

Furthermore, let the initial data  $V_0$  and inhomogeneous term  $F$  as superposition of spherical harmonic functions, i.e.,

$$V_0(r\theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} a_{k,l}(r) Y_{k,l}(\theta) \quad \text{and} \quad F(t, r\theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} b_{k,l}(t, r) Y_{k,l}(\theta).$$

Using the separation of variables, we can write  $V(t, r, \theta)$  as a linear combination of products of radial functions and spherical harmonics,

$$V(t, r, \theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} v_{k,l}(t, r) Y_{k,l}(\theta),$$

where  $v_{k,l}$  is given by

$$\begin{cases} \partial_t v_{k,l} - r^\beta (\partial_{rr} v_{k,l} + \frac{n-1}{r} \partial_r v_{k,l} - \frac{k(k+n-2)}{r^2} v_{k,l}) = b_{k,l}(t, r), \\ v_{k,l}(0, r) = a_{k,l}(r), \end{cases}$$

for each  $k, l \in \mathbb{N}$  and  $1 \leq l \leq d(k)$ . If we denote the operator

$$A_{\mu(k)} := -\partial_r^2 - \frac{n-1}{r} \partial_r + \frac{\mu^2(k) - \lambda^2(n)}{r^2},$$

then, we can rewrite the above equation by the definition of  $A_{\mu(k)}$  as

$$(4) \quad \begin{cases} \partial_t v_{k,l} + r^\beta A_{\mu(k)} v_{k,l} = b_{k,l}(t, r), \\ v_{k,l}(0, r) = a_{k,l}(r). \end{cases}$$

We call equation (4) the  $k$ -th model. In the rest of this section, we skip  $k$  and  $l$  in the notation for convenience's sake by remembering  $\mu = \mu(\beta, k) = \frac{2\mu(k)}{2-\beta}$  and  $\mu(k) = k + \lambda$ .

Next, we introduce the generalized  $\beta$ -Hankel transform and give the mild solution of the  $k$ -th model (4).

**Definition 2.1.** Let  $\beta \in [0, 2)$ ,  $\phi(r)$  and  $\psi(r)$  be integrable functions in  $\mathbb{R}^+$ , we define the generalized  $\beta$ -Hankel transform of  $\phi(r)$  as follows,

$$\mathcal{H}_\mu \phi(\rho) := \int_0^\infty U(r\rho) \phi(r) r^{n-1} dr, \quad U(w) = w^{\frac{2-n-2\beta}{2}} J_\mu\left(\frac{2}{2-\beta} w^{\frac{2-\beta}{2}}\right)$$

and its inversion on  $\psi(\rho)$ ,

$$\mathcal{H}_\mu^{-1}\psi(r) := \int_0^\infty V(r\rho)\psi(\rho)\rho^{n-1}d\rho, \quad V(w) = w^{\frac{2-n}{2}}J_\mu\left(\frac{2}{2-\beta}w^{\frac{2-\beta}{2}}\right),$$

where  $J_\mu(x)$  is the first kind of Bessel function of real order  $\mu = \mu(\beta, k) > -\frac{1}{2}$  defined as

$$J_\mu(r) := \frac{(r/2)^\mu}{\Gamma(\mu + 1/2)\pi^{1/2}} \int_{-1}^1 e^{irt}(1-t^2)^{\mu-1/2}dt.$$

Before going further, we need to prove  $\mathcal{H}_\mu^{-1}$  in Definition 2.1 is the true inverse of  $\mathcal{H}_\mu$ . Given  $\phi \in L(\mathbb{R}^+)$ , we have

$$\begin{aligned} \mathcal{H}_\mu^{-1}\mathcal{H}_\mu\phi(r) &= \int_0^\infty V(r\rho) \int_0^\infty U(s\rho)\phi(s)s^{n-1}ds\rho^{n-1}d\rho \\ &= \int_0^\infty \phi(s)s^{n-1}ds \int_0^\infty V(r\rho)U(s\rho)\rho^{n-1}d\rho. \end{aligned}$$

After a proper scaling calculation, we find

$$\begin{aligned} &\int_0^\infty V(r\rho)U(s\rho)\rho^{n-1}d\rho \\ &= \frac{2}{2-\beta}r^{\frac{2-n}{2}}s^{\frac{2-n-2\beta}{2}} \int_0^\infty J_\mu\left(\frac{2}{2-\beta}r^{\frac{2-\beta}{2}}\rho^{\frac{2-\beta}{2}}\right)J_\mu\left(\frac{2}{2-\beta}s^{\frac{2-\beta}{2}}\rho^{\frac{2-\beta}{2}}\right)\rho^{\frac{2-\beta}{2}}d\rho^{\frac{2-\beta}{2}} \\ &= r^{\frac{\beta-n}{2}}s^{\frac{2-n-2\beta}{2}}\delta\left(\frac{2}{2-\beta}r^{\frac{2-\beta}{2}} - \frac{2}{2-\beta}s^{\frac{2-\beta}{2}}\right), \end{aligned}$$

where  $\delta$  is the delta function. Thus,

$$\begin{aligned} \mathcal{H}_\mu^{-1}\mathcal{H}_\mu\phi(r) &= \int_0^\infty \phi(s)r^{\frac{\beta-n}{2}}s^{\frac{n-\beta}{2}}\delta\left(\frac{2}{2-\beta}r^{\frac{2-\beta}{2}} - \frac{2}{2-\beta}s^{\frac{2-\beta}{2}}\right)d\left(\frac{2}{2-\beta}s^{\frac{2-\beta}{2}}\right) \\ &= \phi(r). \end{aligned}$$

As a result,  $\beta$ -Hankel transform and its inverse are well-defined. We have the following properties for the  $\beta$ -Hankel transform:

**Proposition 2.2.** *Let  $\mathcal{H}_{\mu(\beta,k)}$  and  $A_{\mu(k)}$  be defined as above, then,*

(i)  $\mathcal{H}_\mu$  and  $\mathcal{H}_\mu^{-1}$  are self-adjoint, i.e.,  $\mathcal{H}_\mu = \mathcal{H}_\mu^*$  and  $\mathcal{H}_\mu^{-1} = \mathcal{H}_\mu^{-1*}$

(ii)  $\int_0^\infty \mathcal{H}_\mu^2\phi(\rho)\rho^{\beta+n-1}d\rho = \int_0^\infty \phi^2(r)r^{-\beta+n-1}dr.$

(iii)  $\mathcal{H}_{\mu(\beta,k)}(r^\beta A_{\mu(k)}\phi)(\rho) = \rho^{2-\beta}\mathcal{H}_{\mu(\beta,k)}(\phi)(\rho).$

*Proof.* (i) This is obvious from definition.

(ii) Observe that

$$\mathcal{H}_\mu(r^\beta\phi(r))(\rho) = \rho^{-\beta}\mathcal{H}_\mu^{-1}(\phi(r))(\rho),$$

by combining property (i), one has

$$\begin{aligned} &< \mathcal{H}_\mu\phi(\rho), \rho^\beta\mathcal{H}_\mu\psi(\rho) > = < \phi(r), \mathcal{H}_\mu(\rho^\beta\mathcal{H}_\mu\psi(\rho))(r) > \\ &= < \phi(r), r^{-\beta}\mathcal{H}_\mu^{-1}(\mathcal{H}_\mu\psi(\rho))(r) > = < \phi(r), r^{-\beta}\psi(r) >. \end{aligned}$$

(iii) Using Definition 2.1 and integrating by parts, we have

$$\begin{aligned}
 \mathcal{H}_{\mu(\beta,k)}(r^\beta A_{\mu(k)}\phi)(\rho) &= \int_0^\infty \left(-\partial_r^2 - \frac{n-1}{r}\partial_r + \frac{\mu(k)^2 - \lambda^2}{r^2}\right)\phi(r)U(r\rho)r^{\beta+n-1}dr \\
 &= \int_0^\infty r^{\beta+n-3} \left\{ (-\beta(\beta+n-2) + \mu(k)^2 - \lambda^2) U(r\rho) \right. \\
 (5) \quad &\quad \left. + (-2\beta - n + 1)rsU'(r\rho) - (r\rho)^2U''(r\rho) \right\} \phi(r)dr.
 \end{aligned}$$

It is evident  $U(r\rho) = (r\rho)^{\frac{2-n-2\beta}{2}} J_{\mu(\beta,k)}(\frac{2}{2-\beta}(r\rho)^{\frac{2-\beta}{2}})$ , and it satisfies the following Bessel equation [15]

$$(r\rho)^2U''(r\rho) + (n+2\beta-1)rsU'(r\rho) + \left\{ (r\rho)^{2-\beta} + \left(1-\beta-\frac{n}{2}\right)^2 - \left(\frac{2-\beta}{2}\mu(\beta,k)\right)^2 \right\} U(r\rho) = 0.$$

Recalling  $\mu(\beta,k) = \frac{2}{2-\beta}\mu(k)$ , we find that the terms in braces of (5) is equal to  $(r\rho)^{2-\beta}U(r\rho)$ . Thus,

$$\mathcal{H}_{\mu(\beta,k)}(r^\beta A_{\mu(k)}\phi) = \int_0^\infty r^{\beta+n-3}(r\rho)^{2-\beta}U(r\rho)\phi(r)dr = \rho^{2-\beta}\mathcal{H}_\mu\phi(\rho).$$

□

Further we introduce the  $\beta$ -Hankel convolution operator  $\sharp$ .

**Definition 2.3.** Let  $\alpha = \beta - k$ ,  $U$  and  $V$  be defined as above. We define the Delsarte's kernel:

$$D(x, y, z) := \int_0^\infty \eta^\alpha V(x\eta)U(y\eta)U(z\eta)\eta^{n-1}d\eta,$$

the Hankel translate function:

$$f^*(x, y) := \int_0^\infty f(z)D(x, y, z)z^{n-1}dz,$$

and the  $\beta$ -Hankel convolution operator  $\sharp$ :

$$f\sharp g(x) := \int_0^\infty f^*(x, y)g(y)y^{n-1}dy.$$

From Definition 2.3, we easily get

$$\begin{aligned}
 f\sharp g(x) &= \int_0^\infty f^*(x, y)g(y)y^{n-1}dy \\
 &= \int_0^\infty g(y)y^{n-1}dy \int_0^\infty f(z)D(x, y, z)z^{n-1}dz \\
 &= \mathcal{H}^{-1}(\eta^\alpha \mathcal{H}g(\eta)\mathcal{H}f(\eta))(x),
 \end{aligned}$$

which implies

$$(6) \quad \mathcal{H}(f\sharp g)(\eta) = \eta^\alpha \mathcal{H}g(\eta)\mathcal{H}f(\eta).$$

We summary the properties about Delsarte's kernel  $D(x, y, z)$  in the follow proposition.

**Proposition 2.4.** *The following identity holds for  $D(x, y, z)$  defined above:*

$$D(x, y, z) = \frac{(xyz)^{-\lambda-\beta}x^\beta}{1-\beta/2} \cdot \left[ \left( \frac{2}{2-\beta} \right)^3 (xyz)^{\frac{2-\beta}{2}} \right]^{-\mu} \cdot \frac{2^{\mu-1} \Delta^{2\mu-1}}{\Gamma(\mu + \frac{1}{2}) \Gamma(\frac{1}{2})}$$

where  $\Delta$  is the area of a triangle with sides  $(\frac{2(x)^{\frac{2-\beta}{2}}}{2-\beta}, \frac{2(y)^{\frac{2-\beta}{2}}}{2-\beta}, \frac{2(z)^{\frac{2-\beta}{2}}}{2-\beta})$  if such a triangle exists or zero otherwise. Besides, we have

$$(7) \quad \int_0^\infty x^{k-\beta} D(x, y, z) x^{n-1} dx = \Gamma(\mu + 1)^{-1} (2 - \beta)^{-\mu} (yz)^{k-\beta};$$

$$(8) \quad \int_0^\infty y^k D(x, y, z) y^{n-1} dy = \Gamma(\mu + 1)^{-1} (2 - \beta)^{-\mu} x^k z^{k-\beta};$$

$$(9) \quad \int_0^\infty z^k D(x, y, z) z^{n-1} dz = \Gamma(\mu + 1)^{-1} (2 - \beta)^{-\mu} x^k y^{k-\beta}.$$

*Proof.* Writing  $D(x, y, z)$  in integration, we easily find

$$\begin{aligned} D(x, y, z) &= \int_0^\infty \eta^{\beta-k} V(x\eta) U(y\eta) U(z\eta) \eta^{n-1} d\eta \\ &= \frac{(xyz)^{-\lambda-\beta}x^\beta}{1-\beta/2} \int_0^\infty J_\mu\left(\frac{2(x\eta)^{\frac{2-\beta}{2}}}{2-\beta}\right) J_\mu\left(\frac{2(y\eta)^{\frac{2-\beta}{2}}}{2-\beta}\right) J_\mu\left(\frac{2(z\eta)^{\frac{2-\beta}{2}}}{2-\beta}\right) \frac{d\eta^{\frac{2-\beta}{2}}}{\eta^{(1-\beta/2)(\mu-1)}}. \end{aligned}$$

Thus, the first identity follows from [15]. The proof of the next three integration identities are basically the same, so we only prove (7). According to Definition 2.3, we immediately have:

$$(10) \quad \int_0^\infty U(xs) D(x, y, z) x^{n-1} dx = s^\alpha U(ys) U(zs).$$

By the asymptotic behavior of Bessel function,

$$J_{\nu-\frac{1}{2}}(x) \sim \frac{1}{\Gamma(\nu + \frac{1}{2})} \left(\frac{x}{2}\right)^{\nu-\frac{1}{2}},$$

we get

$$\lim_{s \rightarrow 0} s^\alpha U(xs) = \lim_{s \rightarrow 0} s^\alpha (xs)^{\frac{2-n-2\beta}{2}} J_\mu\left(\frac{2}{2-\beta} (xs)^{\frac{2-\beta}{2}}\right) = \Gamma(\mu + 1)^{-1} (2 - \beta)^{-\mu} x^{k-\beta}.$$

Similarly, one will also find

$$\lim_{s \rightarrow 0} s^{-k} V(ys) = \Gamma(\mu + 1)^{-1} (2 - \beta)^{-\mu} y^k.$$

Multiplying  $s^\alpha$  to (10) and let  $s$  go to 0 on both sides, we derive (7).  $\square$

With this proposition, we can prove the next lemma which is the Young's inequality for  $\beta$ -Hankel convolution.

**Lemma 2.5.** *For the convolution  $\sharp$  defined above, we have:*

$$\left( \int_0^\infty \left| \frac{f \sharp g(x)}{x^k} \right|^a x^{2k+n-1-\beta} dx \right)^{\frac{1}{a}}$$



$$\leq |\Gamma(\mu+1)^{-1}(2-\beta)^{-\mu}| \left( \int_0^\infty \left| \frac{f(z)}{z^k} \right|^b z^{2k+n-1-\beta} dz \right)^{\frac{1}{b}} \left( \int_0^\infty \left| \frac{g(y)}{y^k} \right|^c y^{2k+n-1-\beta} dy \right)^{\frac{1}{c}},$$

where  $1 + \frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ .

*Proof.* We start from the integration of translate function  $f^*(x, y)$ .

$$\begin{aligned} & \int_0^\infty \left| \frac{f^*(x, y)}{y^{k-\beta}} \right|^p y^{2k+n-1-\beta} dy = \int_0^\infty |f^*(x, y)|^p y^{2k+n-1-\beta-p(k-\beta)} dy \\ &= \int_0^\infty \left| \int_0^\infty \frac{f(z)}{z^k} (xy)^k y^{-\beta} \frac{D(x, y, z)}{(xy)^k y^{-\beta}} z^{k+n-1} dz \right|^p y^{2k+n-1-\beta-p(k-\beta)} dy. \end{aligned}$$

From (9) we know

$$\int_0^\infty z^k D(x, y, z) z^{n-1} dz = \Gamma(\mu+1)^{-1} (2-\beta)^{-\mu} x^k y^{k-\beta}.$$

By applying Jensen's inequality on has

$$\begin{aligned} & \left| \int_0^\infty \frac{f(z)}{z^k} (xy)^k y^{-\beta} \frac{D(x, y, z)}{(xy)^k y^{-\beta}} z^{k+n-1} dz \right|^p \\ & \leq (\Gamma(\mu+1)^{-1} (2-\beta)^{-\mu})^{p-1} \int_0^\infty \left| \frac{f(z)}{z^k} (xy)^k y^{-\beta} \right|^p \frac{D(x, y, z)}{(xy)^k y^{-\beta}} z^{k+n-1} dz. \end{aligned}$$

By changing the order of integration again

$$\begin{aligned} & \int_0^\infty \left| \frac{f^*(x, y)}{y^{k-\beta}} \right|^p y^{2k+n-1-\beta} dy \leq (\Gamma(\mu+1)^{-1} (2-\beta)^{-\mu})^{p-1} \\ & \cdot \int_0^\infty \left| \frac{f(z)}{z^k} \right|^p d\sigma(z) \int_0^\infty [(xy)^k y^{-\beta}]^{p-1} D(x, y, z) z^{k+n-1} y^{2k+n-1-\beta-p(k-\beta)} dy \\ &= (\Gamma(\mu+1)^{-1} (2-\beta)^{-\mu})^{p-1} \int_0^\infty \left| \frac{f(z)}{z^k} \right|^p d\sigma(z) x^{k(p-1)} \int_0^\infty D(x, y, z) d\sigma(y) \\ &= (\Gamma(\mu+1)^{-1} (2-\beta)^{-\mu})^p x^{kp} \int_0^\infty \left| \frac{f(z)}{z^k} \right|^p z^{2k+n-1-\beta} dz. \end{aligned}$$

Similarly, one can obtain

$$\int_0^\infty \left| \frac{f^*(x, y)}{x^k} \right|^p x^{2k+n-1-\beta} dx \leq (\Gamma(\mu+1)^{-1} (2-\beta)^{-\mu})^p y^{(k-\beta)p} \int_0^\infty \left| \frac{f(z)}{z^k} \right|^p z^{2k+n-1-\beta} dz.$$

Let  $d\eta(x) = x^{2k+n-1-\beta} dx$ . By Young's inequality ([14]), we have

$$\begin{aligned} |f \sharp g| &= \left| \int_0^\infty f^*(x, y) g(y) y^{n-1} dy \right| = \left| \int_0^\infty \frac{f^*(x, y)}{y^{k-\beta}} \cdot \frac{g(y)}{y^k} \cdot y^{2k+n-1-\beta} dy \right| \\ &\leq \left( \int_0^\infty \left| \frac{f^*(x, y)}{y^{k-\beta}} \right|^p \cdot \left| \frac{g(y)}{y^k} \right|^q d\eta(y) \right)^{\frac{1}{m}} \\ &\quad \cdot \left( \int_0^\infty \left| \frac{f^*(x, y)}{y^{k-\beta}} \right|^p d\eta(y) \right)^{1-\frac{1}{q}} \cdot \left( \int_0^\infty \left| \frac{g(y)}{y^k} \right|^q d\eta(y) \right)^{1-\frac{1}{p}} \\ &= I \cdot II \cdot III. \end{aligned}$$

And we have

$$II^{\frac{q}{q-1}} \leq (\Gamma(\mu+1)^{-1}(2-\beta)^{-\mu})^p x^{kp} \int_0^\infty \left| \frac{f(z)}{z^k} \right|^p d\eta(z)$$

and

$$\begin{aligned} \int_0^\infty \left| \frac{f \sharp g}{x^k} \right|^m d\eta(x) &\leq \int_0^\infty \left( \frac{I \cdot II \cdot III}{x^k} \right)^m d\eta(x) \\ &= \left( \int_0^\infty \left| \frac{g(y)}{y^k} \right|^q d\eta(y) \right)^{\frac{m(p-1)}{p}} \int_0^\infty \left( \int_0^\infty \left| \frac{f^*(x, y)}{y^{k-\beta}} \right|^p \cdot \left| \frac{g(y)}{y^k} \right|^q d\eta(y) \right) \cdot II^m x^{-km} d\eta(x) \\ &\leq (\Gamma(\mu+1)^{-1}(2-\beta)^{-\mu})^{\frac{mp(q-1)}{q}} \left( \int_0^\infty \left| \frac{g(y)}{y^k} \right|^q d\eta(y) \right)^{\frac{(p-1)m}{p}} \left( \int_0^\infty \left| \frac{f(z)}{z^k} \right|^p d\eta(z) \right)^{\frac{m(q-1)}{q}} \\ &\quad \cdot \int_0^\infty \left( \int_0^\infty \left| \frac{f^*(x, y)}{y^{k-\beta}} \right|^p \cdot \left| \frac{g(y)}{y^k} \right|^q d\eta(y) \right) \cdot x^{k\frac{mp(q-1)}{q}} x^{-km} d\eta(x) \\ &= (\Gamma(\mu+1)^{-1}(2-\beta)^{-\mu})^{\frac{pm(q-1)}{q}} \left( \int_0^\infty \left| \frac{g(y)}{y^k} \right|^q d\eta(y) \right)^{\frac{(p-1)m}{p}} \left( \int_0^\infty \left| \frac{f(z)}{z^k} \right|^p d\eta(z) \right)^{\frac{(q-1)m}{q}} \\ &\quad \cdot \int_0^\infty \left| \frac{g(y)}{y^k} \right|^q y^{(\beta-k)p} d\eta(y) \cdot \int_0^\infty \left| \frac{f^*(x, y)}{x^k} \right|^p d\eta(x) \\ &\leq (\Gamma(\mu+1)^{-1}(2-\beta)^{-\mu})^{\frac{mp(q-1)}{q}} \left( \int_0^\infty \left| \frac{g(y)}{y^k} \right|^q d\eta(y) \right)^{\frac{m(p-1)}{p}} \left( \int_0^\infty \left| \frac{f(z)}{z^k} \right|^p d\eta(z) \right)^{\frac{m(q-1)}{q}} \\ &\quad \cdot \int_0^\infty \left| \frac{g(y)}{y^k} \right|^q d\eta(y) \cdot (\Gamma(\mu+1)^{-1}(2-\beta)^{-\mu})^p \int_0^\infty \left| \frac{f(z)}{z^k} \right|^p d\eta(z) \\ &= (\Gamma(\mu+1)^{-1}(2-\beta)^{-\mu})^m \left( \int_0^\infty \left| \frac{g(y)}{y^k} \right|^q d\eta(y) \right)^{\frac{m}{q}} \left( \int_0^\infty \left| \frac{f(z)}{z^k} \right|^p d\eta(z) \right)^{\frac{m}{p}}. \end{aligned}$$

That is,

$$\left( \int_0^\infty \left| \frac{f \sharp g}{x^k} \right|^m d\eta(x) \right)^{\frac{1}{m}} \leq (\Gamma(\mu+1)^{-1}(2-\beta)^{-\mu}) \left( \int_0^\infty \left| \frac{g(y)}{y^k} \right|^q d\eta(y) \right)^{\frac{1}{q}} \left( \int_0^\infty \left| \frac{f(z)}{z^k} \right|^p d\eta(z) \right)^{\frac{1}{p}}.$$

□

Finally, we can derive the integral solution of linear  $k$ -th model (4). Applying the  $\beta$ -Hankel transform, we get

$$\begin{cases} \partial_t \mathcal{H}_\mu v + \rho^{2-\beta} \mathcal{H}_\mu v = \mathcal{H}_\mu b, \\ \mathcal{H}_\mu v(0, \rho) = (\mathcal{H}_\mu a)(\rho). \end{cases}$$

Solving the ODE and further applying  $\mathcal{H}_\mu^{-1}$ , we get its explicit solution formula which can be also represented in terms of Hankel convolution:

$$\begin{aligned} v(t, r) &= \mathcal{H}_\mu^{-1}[\exp(-\rho^{2-\beta}t)\mathcal{H}_\mu a](r) + \mathcal{H}_\mu^{-1}\left[\int_0^t \exp(-\rho^{2-\beta}(t-\tau))\mathcal{H}_\mu b(\rho, \tau)d\tau\right](r) \\ &= \mathcal{H}_\mu^{-1}\left[\frac{\exp(-\rho^{2-\beta}t)}{\rho^{\beta-k}}\right](r)\sharp a(r) + \int_0^t \mathcal{H}_\mu^{-1}\left[\frac{\exp(-\rho^{2-\beta}(t-\tau))}{\rho^{\beta-k}}\right](r)\sharp b(r, \tau)d\tau. \end{aligned}$$

Define the solution's kernel  $K_\mu(r, t)$  by

$$K_\mu(r, t) := \mathcal{H}_\mu^{-1}\left[\frac{\exp(-\rho^{2-\beta}t)}{\rho^{\beta-k}}\right](r)$$

and the solution semigroup  $S_\mu(t) (\triangleq e^{r^\beta A_{\mu(k)}t})$  by

$$S_\mu(t)f := K_\mu(r, t) \sharp f,$$

then, the solution can be written in a simple form

$$(11) \quad v(t, r) = S_\mu(t)a(r) + \int_0^t S_\mu(t-\tau)b(\tau, r)d\tau.$$

### 3. SPACE-TIME ESTIMATES FOR THE LINEAR $k$ -TH MODEL

In this section, we analyse the kernel  $K_\mu(r, t)$  and the semigroup  $S_\mu(t)$ . After that we discuss the space-time estimates of solution to the  $k$ -th model.

We start from the definition of  $K_\mu(r, t)$ .

$$\begin{aligned} K_\mu(r, t) &= \mathcal{H}_\mu^{-1}\left(\frac{\exp(-\rho^{2-\beta}t)}{\rho^{\beta-k}}\right)(r) \\ &= \int_0^\infty (r\rho)^{-\lambda} J_\mu\left(\frac{2}{2-\beta}(r\rho)^{\frac{2-\beta}{2}}\right) \frac{\exp(-\rho^{2-\beta}t)}{\rho^{\beta-k}} \rho^{n-1} d\rho \\ &= \frac{2r^{-\lambda}}{2-\beta} \int_0^\infty \exp(-\rho^{2-\beta}t) J_\mu\left(\frac{2r^{\frac{2-\beta}{2}}}{2-\beta} \rho^{\frac{2-\beta}{2}}\right) (\rho^{\frac{2-\beta}{2}})^{\mu+1} d\rho^{\frac{2-\beta}{2}} \\ (12) \quad &= \{(2-\beta)t\}^{-\mu-1} \exp\left(-\frac{r^{2-\beta}}{(2-\beta)^2 t}\right) r^k. \end{aligned}$$

The last equality is due to the identity from [15]

$$(13) \quad \int_0^\infty J_\nu(at) \exp(-p^2 t^2) t^{\nu+1} dt = \frac{a^\nu}{(2p^2)^{\nu+1}} \exp\left(-\frac{a^2}{4p^2}\right).$$

For the solution semigroup  $S_\mu(t)$ , by changing the order of integration, we get

$$(14) \quad S_\mu(t)a(r) = \int_0^\infty \tilde{K}(\rho, r, t) a(\rho) \rho^{n-1} d\rho,$$

where  $\tilde{K}(\rho, r, t) :=$

$$\frac{2}{2-\beta} r^{-\lambda} \rho^{-\lambda-\beta} \int_0^\infty J_\mu\left(\frac{2}{2-\beta}(r\xi)^{\frac{2-\beta}{2}}\right) J_\mu\left(\frac{2}{2-\beta}(\rho\xi)^{\frac{2-\beta}{2}}\right) \exp(-\xi^{2-\beta}t) \xi^{\frac{2-\beta}{2}} d\xi^{\frac{2-\beta}{2}}.$$

This integral is equivalent to the Weber's second exponential integral after a proper scaling calculation (See [15] p395). The convergence is secured by  $\exp(-\xi^{2-\beta}t)$  and  $\mu > 0$ . Moreover, we have

$$\tilde{K}(r, \rho, t) = \frac{r^{-\lambda} \rho^{-\lambda-\beta}}{(2-\beta)t} \cdot \exp\left\{-\frac{1}{t(2-\beta)^2}(r^{2-\beta} + \rho^{2-\beta})\right\} \cdot I_\mu\left(\frac{2(r\rho)^{\frac{2-\beta}{2}}}{(2-\beta)^2 t}\right),$$

where

$$I_\mu(x) = i^{-\mu} J_\mu(ix)$$

stands for the modified Bessel function. Consequently,

$$S_\mu(t)a(r) = \frac{2r^{-\lambda} \exp(-\frac{r^{2-\beta}}{t(2-\beta)^2})}{(2-\beta)^2 t} \cdot \int_0^\infty i^{-\mu} J_\mu(\frac{2ir^{\frac{2-\beta}{2}}}{(2-\beta)^2 t} \rho^{\frac{2-\beta}{2}}) \exp(-\frac{\rho^{2-\beta}}{t(2-\beta)^2}) a(\rho) \rho^{\frac{2-\beta}{2}(\mu+1)-k} d\rho^{\frac{2-\beta}{2}}.$$

If we apply the Hölder's inequality directly and recall the identity (13) again, we obtain:

$$\begin{aligned} \left| \frac{S_\mu(t)a(r)}{r^k} \right| &\leq \left\| \frac{a(\rho)}{\rho^k} \right\|_{L^\infty} \frac{2r^{-\lambda-k} \exp(-\frac{r^{2-\beta}}{t(2-\beta)^2})}{(2-\beta)^2 t} \cdot \\ &\left| \int_0^\infty i^{-\mu} J_\mu(\frac{2ir^{\frac{2-\beta}{2}}}{(2-\beta)^2 t} \rho^{\frac{2-\beta}{2}}) \exp(-\frac{\rho^{2-\beta}}{t(2-\beta)^2}) \rho^{\frac{2-\beta}{2}(\mu+1)} d\rho^{\frac{2-\beta}{2}} \right|. \end{aligned}$$

That is

$$(15) \quad \left\| \frac{S_\mu(t)a(r)}{r^k} \right\|_{L^\infty(r)} \leq \left\| \frac{a(\rho)}{\rho^k} \right\|_{L^\infty(\rho)}.$$

In fact, by applying Young's inequality in Lemma 2.5, we have the following  $L^p - L^q$  estimates for homogeneous part of the solution.

**Lemma 3.1.** *Let  $d\eta(r) = r^{2k+n-1-\beta} dr$  and  $1 \leq q \leq p \leq \infty$ . Then  $S_\mu(t)a(r)$  satisfies the following estimates,*

$$(16) \quad \left\| \frac{S_\mu(t)a(r)}{r^k} \right\|_{L^p_{d\eta(r)}} \leq C(\beta, \mu, p, q) t^{\frac{2k+n-\beta}{2-\beta}(\frac{1}{p}-\frac{1}{q})} \left\| \frac{a(r)}{r^k} \right\|_{L^q_{d\eta(r)}},$$

where constant  $C(\beta, \mu, p, q)$  is independent of  $k$ .

*Proof.* Since  $S_\mu(t)a(r) = K_\mu(r, t) \sharp a(r)$  and by recalling (12) we find that,

$$\begin{aligned} \left\| \frac{K_\mu(r, t)}{r^k} \right\|_{L^m_{d\eta(r)}} &= \{(2-\beta)t\}^{-\mu-1} \left( \int_0^\infty \exp(-\frac{mr^{2-\beta}}{(2-\beta)^2 t}) r^{2k+n-1-\beta} dr \right)^{\frac{1}{m}} \\ &\leq (2-\beta)^{\frac{4k+2n-\beta-2}{(2-\beta)m}-\mu-1} m^{-\frac{2k+n-\beta}{(2-\beta)m}} \Gamma(\mu+1)^{\frac{1}{m}} t^{\frac{2k+n-\beta}{2-\beta}(\frac{1}{m}-1)}. \end{aligned}$$

Hence, the Young's inequality gives the desired inequality (16) by taking  $1 + \frac{1}{p} = \frac{1}{m} + \frac{1}{q}$  and

$$(17) \quad C(\beta, \mu, p, q) = [(2-\beta)^{(2\mu+1)} \Gamma(\mu+1)]^{\frac{1}{p}-\frac{1}{q}} m^{-\frac{2k+n-\beta}{(2-\beta)m}}.$$

Since  $\frac{1}{p} - \frac{1}{q} \leq 0$  and  $(2-\beta)^{(2\mu+1)} \Gamma(\mu+1)$  goes to infinity as  $k \rightarrow \infty$ , we find that the constant  $C(\beta, \mu, p, q)$  is independent of  $k$ .  $\square$

It is remarkable that (16) generalizes the result of (15) by taking  $m = 1, p = q$ ,

$$\left\| \frac{S_\mu(t)a(r)}{r^k} \right\|_{L^p_{d\eta(r)}} \leq \left\| \frac{a(r)}{r^k} \right\|_{L^p_{d\eta(r)}}.$$

At the moment we state the space-time estimates for the homogeneous part of solution  $v$  given in (11). Its proof can be made by following [4] (see also [9]).

**Lemma 3.2.** (i) Let  $\psi$  satisfy  $\|\frac{\psi}{r^k}\|_{L^q_{d\eta(r)}} < \infty$  and  $(m, p, q)$  be any admissible triplet. Then,  $\frac{S_\mu(t)\psi}{r^k} \in L^m(I; L^p_{d\eta(r)}(\mathbb{R}^+)) \cap C_b(I; L^q_{d\eta(r)}(\mathbb{R}^+))$  with the estimate

$$(18) \quad \left\| \frac{S_\mu(t)\psi}{r^k} \right\|_{L^m(I; L^p_{d\eta(r)})} \leq C \left\| \frac{\psi}{r^k} \right\|_{L^q_{d\eta(r)}}$$

for  $0 < T \leq \infty$ , where  $C$  is a positive constant independent of  $k$ .

(ii) Let  $\psi$  satisfy  $\|\frac{\psi}{r^k}\|_{L^q_{d\eta(r)}} < \infty$  and  $(m, p, q)$  be any generalized admissible triplet. Then,  $\frac{S_\mu(t)\psi}{r^k} \in \mathcal{C}_m(I; L^p_{d\eta(r)}(\mathbb{R}^+)) \cap C_b(I; L^q_{d\eta(r)}(\mathbb{R}^+))$  with the estimate

$$(19) \quad \left\| \frac{S_\mu(t)\psi}{r^k} \right\|_{\mathcal{C}_m(I; L^p_{d\eta(r)})} \leq C \left\| \frac{\psi}{r^k} \right\|_{L^q_{d\eta(r)}}$$

for  $0 < T \leq \infty$ , where  $C$  is a positive constant independent of  $k$ .

Hereafter, for a Banach space  $X$ , we denote by  $C_b(I; X)$  the space of bounded continuous functions from  $I$  to  $X$ .

*Proof.* The statement (ii) follows easily from Lemma 3.1. It suffices to prove (i). For the case  $p = q$  and  $m = \infty$ , the space-time estimate is true from (16). We now consider the case  $p > q$ . Assume  $(\tilde{m}, \tilde{p}, \tilde{q})$  be an admissible triplet and define the operator

$$U\psi = \left\| \frac{S_\mu(t)\psi}{r^k} \right\|_{L^{\tilde{p}}_{d\eta(r)}}$$

from an weighted  $L^q$  space to functions on  $[0, T)$ . As the Young's inequality (16) gives

$$U\psi \leq Ct^{\frac{-n+\beta-2k}{2-\beta}(\frac{1}{\tilde{q}}-\frac{1}{\tilde{p}})} \left\| \frac{\psi}{r^k} \right\|_{L^{\tilde{q}}_{d\eta(r)}} = Ct^{-\frac{1}{\tilde{m}}} \left\| \frac{\psi}{r^k} \right\|_{L^{\tilde{q}}_{d\eta(r)}}.$$

It is easy to see that

$$\begin{aligned} m(t : |U\psi| > \tau) &\leq m\{t : Ct^{-\frac{1}{\tilde{m}}} \left\| \frac{\psi}{r^k} \right\|_{L^{\tilde{q}}_{d\eta(r)}} > \tau\} \\ &= m\{t : t < \left( \frac{C \left\| \frac{\psi}{r^k} \right\|_{L^{\tilde{q}}_{d\eta(r)}}}{\tau} \right)^{\tilde{m}}\} \\ &\leq \left( \frac{C \left\| \frac{\psi}{r^k} \right\|_{L^{\tilde{q}}_{d\eta(r)}}}{\tau} \right)^{\tilde{m}}, \end{aligned}$$

which implies that  $U$  is a weak type  $(\tilde{q}, \tilde{m})$  operator. On the other hand,  $U$  is sub-additive and satisfies that:

$$U\psi = \left\| \frac{S_\mu(t)\psi}{r^k} \right\|_{L^{\tilde{p}}_{d\eta(r)}} \leq C \left\| \frac{\psi}{r^k} \right\|_{L^{\tilde{p}}_{d\eta(r)}}$$

for  $q \leq \tilde{p} \leq \infty$ , which means that  $U$  is a  $(\tilde{p}, \infty)$  type operator. For any given admissible triplet  $(m, p, q)$ , we choose proper  $(\tilde{m}, \tilde{p}, \tilde{q})$  and  $\theta$  such that

$$\begin{aligned} \frac{1}{q} &= \frac{\theta}{\tilde{q}} + \frac{1-\theta}{\tilde{p}}, \\ \frac{1}{m} &= \frac{\theta}{\tilde{m}} + \frac{1-\theta}{\infty}, \\ p &= \tilde{p}. \end{aligned}$$

Then, the operator  $U$  is of type  $(q, m)$  by the Marcinkiewicz interpolation theorem, i.e.,

$$\|U\psi\|_{L^m} \leq C \left\| \frac{\psi}{r^k} \right\|_{L^q_{d\eta(r)}},$$

which is just the desired result (18).  $\square$

**Remark 3.3.** Let  $m = \infty$  and  $p = q = 2$ , from (18) we derive

$$\forall k \geq 0 \quad \sup_{t>0} \int_0^\infty |S_\mu(t)\psi_k|^2 r^{n-1-\beta} dr \leq C \int_0^\infty |\psi_k|^2 r^{n-1-\beta} dr.$$

Summing over with  $k$  and  $l$ , we obtain:

$$\sup_{t>0} \int \sum_{k,l} |S_\mu(t)\psi_k|^2 r^{n-1-\beta} dr \leq C \int_0^\infty \sum_{k,l} |\psi_k|^2 r^{n-1-\beta} dr.$$

That is

$$\|v(t, r, \theta)\|_{L_t^\infty L^2(r^{n-1-\beta} dr) L_\theta^2} \leq C \|u_0\|_{L^2(r^{n-1-\beta} dr) L_\theta^2}.$$

Now, we move to the nonhomogeneous part of solution. From here and following, we denote

$$\mathbb{G}(f)(t, r) := \int_0^t S_\mu(t - \tau) f(\tau, r) d\tau$$

and

$$\gamma = \frac{n - \beta + 2k}{2 - \beta}.$$

As matter of fact, we have the following space-time estimates in  $\mathcal{L}$  space framework.

**Lemma 3.4.** For  $b > 0$  and  $T > 0$ , let  $q_0 = b\gamma$ ,  $I = [0, T)$ . Assume  $q \geq q_0 > 1$  and  $(m, p, q)$  is an admissible triplet satisfying  $p > b + 1$ .

(i) If  $\frac{f}{r^k} \in L^{\frac{m}{b+1}}(I; L^{\frac{p}{b+1}}_{d\eta(r)})$ , then,

$$\left\| \frac{\mathbb{G}f}{r^k} \right\|_{L^\infty(I; L^q_{d\eta(r)})} \leq CT^{1-\frac{b\gamma}{q}} \left\| \frac{f}{r^k} \right\|_{L^{\frac{m}{b+1}}(I; L^{\frac{p}{b+1}}_{d\eta(r)})}$$

for  $p \leq q(1 + b)$  and

$$\left\| \frac{\mathbb{G}f}{r^k} \right\|_{L^\infty(I; L^q_{d\eta(r)})} \leq CT^{1-\frac{b\gamma}{q}} \left\| \left| \frac{f}{r^k} \right|^{\frac{1}{b+1}} \right\|_{L^\infty(I; L^q_{d\eta(r)})}^{\theta(b+1)} \left\| \left| \frac{f}{r^k} \right|^{\frac{1}{b+1}} \right\|_{L^m(I; L^p_{d\eta(r)})}^{(1-\theta)(b+1)}$$

for  $p > q(1 + b)$ , where  $\theta = \frac{p-q(b+1)}{(b+1)(p-q)}$ .

(ii) If  $\frac{f}{r^k} \in L^{\frac{m}{b+1}}(I; L^{\frac{p}{b+1}}_{d\eta(r)})$ , then,

$$\left\| \frac{\mathbb{G}f}{r^k} \right\|_{L^m(I; L^p_{d\eta(r)})} \leq CT^{1-\frac{b\gamma}{q}} \left\| \frac{f}{r^k} \right\|_{L^{\frac{m}{b+1}}(I; L^{\frac{p}{b+1}}_{d\eta(r)})}$$

for  $p \leq q(1 + b)$  and

$$\left\| \frac{\mathbb{G}f}{r^k} \right\|_{L^m(I; L^p_{d\eta(r)})} \leq CT^{1-\frac{b\gamma}{q}} \left\| \left| \frac{f}{r^k} \right|^{\frac{1}{b+1}} \right\|_{L^\infty(I; L^q_{d\eta(r)})}^{\theta(b+1)} \left\| \left| \frac{f}{r^k} \right|^{\frac{1}{b+1}} \right\|_{L^m(I; L^p_{d\eta(r)})}^{(1-\theta)(b+1)}$$

for  $p > q(1 + b)$ , where  $\theta$  is the same as in (i).

*Proof.* First we prove (i). Consider the case when  $p \leq q(b+1)$ . Using Lemma 3.1 and Hölder's inequality on  $t$ , one has

$$\begin{aligned} \left\| \frac{\mathbb{G}f}{r^k} \right\|_{L^\infty(I; L^q_{d\eta(r)})} &\leq C \int_0^t (t-\tau)^{-\gamma(\frac{b+1}{p}-\frac{1}{q})} \left\| \frac{f(\tau, r)}{r^k} \right\|_{L^{\frac{p}{b+1}}_{d\eta(r)}} d\tau \\ &\leq C \left( \int_0^t (t-\tau)^{-\gamma(\frac{b+1}{p}-\frac{1}{q})\chi} d\tau \right)^{\frac{1}{\chi}} \left\| \frac{f(\tau, r)}{r^k} \right\|_{L^{\frac{m}{b+1}}(I; L^{\frac{p}{b+1}}_{d\eta(r)})} \\ &\leq CT^{1-\frac{b\gamma}{q}} \left\| \frac{f}{r^k} \right\|_{L^{\frac{m}{b+1}}(I; L^{\frac{p}{b+1}}_{d\eta(r)})}, \end{aligned}$$

where  $\frac{1}{\chi} = 1 - \frac{b+1}{m}$  and  $C = C(\mu, p, q, b)$ . For the case  $p > q(b+1)$ , by means of the Riesz interpolation theorem and Hölder's inequality, we have:

$$\begin{aligned} \left\| \frac{\mathbb{G}f}{r^k} \right\|_{L^\infty(I; L^q_{d\eta(r)})} &\leq \int_0^t \left\| \left| \frac{f}{r^k} \right|^{\frac{1}{b+1}} \right\|_{L^{q(b+1)}_{d\eta(r)}}^{b+1} d\tau \\ &\leq C \int_0^t \left\| \left| \frac{f(\tau, r)}{r^k} \right|^{\frac{1}{b+1}} \right\|_{L^q_{d\eta(r)}}^{(b+1)\theta} \left\| \left| \frac{f(\tau, r)}{r^k} \right|^{\frac{1}{b+1}} \right\|_{L^p_{d\eta(r)}}^{(b+1)(1-\theta)} d\tau \\ &\leq CT^{1-\frac{(b+1)((1-\theta))}{m}} \left\| \left| \frac{f(\tau, r)}{r^k} \right|^{\frac{1}{b+1}} \right\|_{C(I; L^q_{d\eta(r)})}^{(b+1)\theta} \left\| \left| \frac{f(\tau, r)}{r^k} \right|^{\frac{1}{b+1}} \right\|_{L^m(I; L^p_{d\eta(r)})}^{(b+1)(1-\theta)} \\ &= CT^{1-\frac{b\gamma}{q}} \left\| \left| \frac{f(\tau, r)}{r^k} \right|^{\frac{1}{b+1}} \right\|_{C(I; L^q_{d\eta(r)})}^{(b+1)\theta} \left\| \left| \frac{f(\tau, r)}{r^k} \right|^{\frac{1}{b+1}} \right\|_{L^m(I; L^p_{d\eta(r)})}^{(b+1)(1-\theta)}, \end{aligned}$$

where  $\theta$  satisfies

$$\frac{1}{q(b+1)} = \frac{\theta}{q} + \frac{1-\theta}{p},$$

and

$$1 = \frac{(1+b)(1-\theta)}{m} + \frac{1}{\chi}.$$

We now prove (ii). For the case  $p \leq q(b+1)$ , by Lemma 3.1 and Young's inequality on  $t$ , one has

$$\begin{aligned} \left\| \frac{\mathbb{G}f}{r^k} \right\|_{L^m(I; L^p_{d\eta(r)})} &\leq C \left\| \int_0^t (t-\tau)^{-\gamma(\frac{b+1}{p}-\frac{1}{p})} \left\| \frac{f(\tau, r)}{r^k} \right\|_{L^{\frac{p}{b+1}}_{d\eta(r)}} d\tau \right\|_{L^m} \\ &\leq C \left( \int_0^T \tau^{-\frac{b\gamma}{p}\chi} d\tau \right)^{\frac{1}{\chi}} \left\| \frac{f(\tau, r)}{r^k} \right\|_{L^{\frac{m}{b+1}}(I; L^{\frac{p}{b+1}}_{d\eta(r)})} \\ &\leq CT^{1-\frac{b\gamma}{q}} \left\| \frac{f}{r^k} \right\|_{L^{\frac{m}{b+1}}(I; L^{\frac{p}{b+1}}_{d\eta(r)})}, \end{aligned}$$

where  $1 + \frac{1}{m} = \frac{1}{\chi} + \frac{b+1}{m}$ . For the case  $p > q(b+1)$ , by a similar manner as the proof of (i), one has

$$\left\| \frac{\mathbb{G}f}{r^k} \right\|_{L^m(I; L^p_{d\eta(r)})} \leq C \left\| \int_0^t (t-\tau)^{-\gamma(\frac{1}{q}-\frac{1}{p})} \left\| \left| \frac{f}{r^k} \right|^{\frac{1}{b+1}} \right\|_{L^{q(b+1)}_{d\eta(r)}}^{b+1} d\tau \right\|_{L^m}$$

$$\begin{aligned}
&\leq C \left\| \int_0^t (t-\tau)^{-\gamma(\frac{1}{q}-\frac{1}{p})} \left\| \left| \frac{f(\tau, r)}{r^k} \right|^{\frac{1}{b+1}} \right\|_{L_{d\eta(r)}^q}^{(b+1)\theta} \left\| \left| \frac{f(\tau, r)}{r^k} \right|^{\frac{1}{b+1}} \right\|_{L_{d\eta(r)}^p}^{(b+1)(1-\theta)} d\tau \right\|_{L^m} \\
&\leq C \left( \int_0^T \tau^{-\gamma(\frac{1}{q}-\frac{1}{p})\chi} d\tau \right)^{\frac{1}{\chi}} \left\| \left| \frac{f(\tau, r)}{r^k} \right|^{\frac{1}{b+1}} \right\|_{C(I; L_{d\eta(r)}^q)}^{(b+1)\theta} \left\| \left| \frac{f(\tau, r)}{r^k} \right|^{\frac{1}{b+1}} \right\|_{L^m(I; L_{d\eta(r)}^p)}^{(b+1)(1-\theta)} \\
&\leq CT^{1-\frac{b\gamma}{q}} \left\| \left| \frac{f(\tau, r)}{r^k} \right|^{\frac{1}{b+1}} \right\|_{C(I; L_{d\eta(r)}^q)}^{(b+1)\theta} \left\| \left| \frac{f(\tau, r)}{r^k} \right|^{\frac{1}{b+1}} \right\|_{L^m(I; L_{d\eta(r)}^p)}^{(b+1)(1-\theta)},
\end{aligned}$$

where  $\theta$  and  $\chi$  satisfy

$$\frac{1}{q(b+1)} = \frac{\theta}{q} + \frac{1-\theta}{p}, \quad 1 + \frac{1}{m} = \frac{(b+1)(1-\theta)}{m} + \frac{1}{\chi}$$

with  $q < q(1+b) < p$ .  $\square$

In fact, concerning the nonhomogeneous part, Lemma 3.4 has its counterpart in the  $\mathcal{C}$  space framework. The estimates can be proved by following [9]. We state the result here and leave the proof to readers.

**Lemma 3.5.** *For  $b > 0$  and  $T > 0$ , let  $\gamma = \frac{n-\beta+2k}{2-\beta}$ ,  $q_0 = b\gamma$ ,  $I = [0, T]$ . Assume  $q \geq q_0 > 1$  and  $(m, p, q)$  is an admissible triplet satisfying  $p > b+1$ .*

(i) *If  $\frac{f}{r^k} \in \mathcal{C}_{\frac{m}{b+1}}(I; L_{d\eta(r)}^{\frac{p}{b+1}})$ , then,*

$$\left\| \frac{\mathbb{G}f}{r^k} \right\|_{L^\infty(I; L_{d\eta(r)}^q)} \leq CT^{1-\frac{b\gamma}{q}} \left\| \frac{f}{r^k} \right\|_{\mathcal{C}_{\frac{m}{b+1}}(I; L_{d\eta(r)}^{\frac{p}{b+1}})}$$

for  $p \leq q(1+b)$  and

$$\left\| \frac{\mathbb{G}f}{r^k} \right\|_{L^\infty(I; L_{d\eta(r)}^q)} \leq CT^{1-\frac{b\gamma}{q}} \left\| \left| \frac{f}{r^k} \right|^{\frac{1}{b+1}} \right\|_{L^\infty(I; L_{d\eta(r)}^q)}^{\theta(b+1)} \left\| \left| \frac{f}{r^k} \right|^{\frac{1}{b+1}} \right\|_{\mathcal{C}_m(I; L_{d\eta(r)}^p)}^{(1-\theta)(b+1)}$$

for  $p > q(1+b)$ , where  $\theta = \frac{p-q(b+1)}{(b+1)(p-q)}$ .

(ii) *If  $\frac{f}{r^k} \in \mathcal{C}_{\frac{m}{b+1}}(I; L_{d\eta(r)}^{\frac{p}{b+1}})$ , then,*

$$\left\| \frac{\mathbb{G}f}{r^k} \right\|_{\mathcal{C}_m(I; L_{d\eta(r)}^p)} \leq CT^{1-\frac{b\gamma}{q}} \left\| \frac{f}{r^k} \right\|_{\mathcal{C}_{\frac{m}{b+1}}(I; L_{d\eta(r)}^{\frac{p}{b+1}})}$$

for  $p \leq q(1+b)$  and

$$\left\| \frac{\mathbb{G}f}{r^k} \right\|_{\mathcal{C}_m(I; L_{d\eta(r)}^p)} \leq CT^{1-\frac{b\gamma}{q}} \left\| \left| \frac{f}{r^k} \right|^{\frac{1}{b+1}} \right\|_{L^\infty(I; L_{d\eta(r)}^q)}^{\theta(b+1)} \left\| \left| \frac{f}{r^k} \right|^{\frac{1}{b+1}} \right\|_{\mathcal{C}_m(I; L_{d\eta(r)}^p)}^{(1-\theta)(b+1)}$$

for  $p > q(1+b)$ , where  $\theta$  is the same as in (i).

#### 4. PROOF OF THEOREM 1.4

Following the similar procedure as the linear  $k$ -th model, we also obtain the corresponding integral equation of (2):

$$(20) \quad u(t, r) = \mathcal{T}(u) := S_\mu(t)u_0(r) + \int_0^t S_\mu(t-\tau)F(u(\tau, r))d\tau$$



$$= S_\mu u_0(t, r) + \mathbb{G}(F(u(\tau, r))).$$

We call the solution of the integral form (20) the mild solution.

$$\begin{aligned} \|u(t, r)\|_{X(I)} &\leq \|S_\mu u_0(t, r)\|_{X(I)} + \|\mathbb{G}(F(u(\tau, r)))\|_{X(I)} \\ &= I + II. \end{aligned}$$

Indeed, we have

$$(21) \quad I \leq \|S_\mu(t)u_0(r)\|_{L^m(I; L_{d\eta}^p(\mathbb{R}^+))} + \|S_\mu(t)u_0(r)\|_{L^\infty(I; L_{d\eta}^q(\mathbb{R}^+))} \leq C_1 \|u_0\|_{L_{d\eta}^q}$$

and

$$\begin{aligned} II &= \left\| \int_0^t S_\mu(t-\tau) |u|^b u d\tau \right\|_{L^m(I; L_{d\eta}^p(\mathbb{R}^+))} + \left\| \int_0^t S_\mu(t-\tau) |u|^b u d\tau \right\|_{L^\infty(I; L_{d\eta}^q(\mathbb{R}^+))} \\ &\leq \begin{cases} CT^{1-\frac{b\gamma}{q}} \| |u|^b u \|_{L^{\frac{m}{b+1}}(I; L_{d\eta}^{\frac{p}{b+1}})}, & \text{for } b+1 < p \leq q(b+1), \\ CT^{1-\frac{b\gamma}{q}} \|u\|_{L^\infty(I; L_{d\eta}^q)}^{\theta(b+1)} \|u\|_{L^m(I; L_{d\eta}^p)}^{(1-\theta)(b+1)}, & \text{for } p > q(b+1), \end{cases} \\ (22) \quad &\leq C_2 T^{1-\frac{b\gamma}{q}} \|u\|_{X(I)}^{b+1}. \end{aligned}$$

Combining the estimates for  $I$  and  $II$ , we derive

$$(23) \quad \|\mathcal{T}u\|_{X(I)} \leq C_1 \|u_0\|_{L_{d\eta}^q} + C_2 T^{1-\frac{b\gamma}{q}} \|u\|_{X(I)}^{b+1}.$$

Moreover,

$$\begin{aligned} \|T(u-v)\|_{X(I)} &= \left\| \int_0^t S_\mu(t-\tau) (|u|^b u - |v|^b v) d\tau \right\|_{X(I)} \\ &\leq \left\| \int_0^t S_\mu(t-\tau) (|u|^b + |v|^b) |u-v| d\tau \right\|_{X(I)} \\ &\leq \left\| \int_0^t S_\mu(t-\tau) (|u|^b |u-v|) d\tau \right\|_{X(I)} + \left\| \int_0^t S_\mu(t-\tau) (|v|^b |u-v|) d\tau \right\|_{X(I)}. \end{aligned}$$

At the same time, we obtain

$$\begin{aligned} &\left\| \int_0^t S_\mu(t-\tau) (|u|^b |u-v|) d\tau \right\|_{X(I)} \\ &\leq \begin{cases} CT^{1-\frac{b\gamma}{q}} \| |u|^b (u-v) \|_{L^{\frac{m}{b+1}}(I; L_{d\eta}^{\frac{p}{b+1}})}, & \text{for } b+1 < p \leq q(b+1), \\ CT^{1-\frac{b\gamma}{q}} \| |u|^{\frac{b}{b+1}} (u-v)^{\frac{1}{b+1}} \|_{L^\infty(I; L_{d\eta}^q)}^{\theta(b+1)} \| |u|^{\frac{b}{b+1}} (u-v)^{\frac{1}{b+1}} \|_{L^m(I; L_{d\eta}^p)}^{(1-\theta)(b+1)}, & \text{for } p > q(b+1), \end{cases} \\ &\leq \begin{cases} CT^{1-\frac{b\gamma}{q}} \|u^b\|_{L^{\frac{m}{b}}(I; L_{d\eta}^{\frac{p}{b}})} \|u-v\|_{L^m(I; L_{d\eta}^p)}, & \text{for } b+1 < p \leq q(b+1), \\ CT^{1-\frac{b\gamma}{q}} \|u\|_{L^\infty(I; L_{d\eta}^q)}^{\theta b} \|u-v\|_{L^\infty(I; L_{d\eta}^q)}^\theta \|u\|_{L^m(I; L_{d\eta}^p)}^{(1-\theta)b} \|u-v\|_{L^m(I; L_{d\eta}^p)}^{(1-\theta)}, & \text{for } p > q(b+1), \end{cases} \\ &\leq C_2 T^{1-\frac{b\gamma}{q}} \|u\|_{X(I)}^b \|u-v\|_{X(I)}, \end{aligned}$$

and

$$\left\| \int_0^t S_\mu(t-\tau) (|v|^b |u-v|) d\tau \right\|_{X(I)} \leq C_2 T^{1-\frac{b\gamma}{q}} \|v\|_{X(I)}^b \|u-v\|_{X(I)}.$$

Thus, we have

$$(24) \quad \|\mathcal{T}(u - v)\|_{X(I)} \leq C_2 T^{1-\frac{b\gamma}{q}} (\|u\|_{X(I)}^b + \|v\|_{X(I)}^b) \|u - v\|_{X(I)}.$$

Now we define the metric space as follows,

$$X_p^{Sol}(I) = \left\{ u \in X(I) \left| \|u\|_{X(I)} \leq 2C_1 \|u_0\|_{L_{d\eta}^q}, (2C_1)^b C_2 T^{1-\frac{b\gamma}{q}} \|u_0\|_{L_{d\eta}^q}^b \leq \frac{1}{2} \right. \right\}.$$

The estimates of (23) and (24) imply that  $\mathcal{T}u$  is a contraction mapping from  $X_p^{Sol}$  to itself. We obtained the results (i) and (ii) by applying the Banach contraction mapping principle. Concerning (iii), by standard argument, one is able to show

$$\lim_{t \rightarrow T^*} \|u(t)\|_{L_{d\eta}^q} = \infty.$$

Meanwhile, for arbitrary  $t < s < T^*$  with  $\|u(t)\|_{L_{d\eta}^q} < \infty$ , by following a similar procedure as above, one can find the unique solution in

$$X_p^{Sol}([t, s]) = \left\{ u \in X([t, s]) \left| \|u\|_{X([t, s])} \leq 2C_1 \|u(t)\|_{L_{d\eta}^q}, (2C_1)^b C_2 |s - t|^{1-\frac{b\gamma}{q}} \|u(t)\|_{L_{d\eta}^q}^b \leq \frac{1}{2} \right. \right\}.$$

Thus, there exists  $\varepsilon_0 > 0$  such that

$$\varepsilon_0 \leq (2C_1)^b C_2 |s - t|^{1-\frac{b\gamma}{q}} \|u(t)\|_{L_{d\eta}^q}^b \leq \frac{1}{2},$$

which gives

$$\|u(t)\|_{L_{d\eta}^q} \geq \frac{C\varepsilon_0}{(T^* - t)^{\frac{1}{b} - \frac{\gamma}{q}}}.$$

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